Beta Decay in Stellar Interiors*

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A study is made of the temperature and density dependence of beta-decay rates as they are affected by electron capture from continuum orbits, the absence of atomic binding energies, screening, and the exclusion principle. The rate of allowed electron capture from continuum orbits in a Fermi gas is calculated using the *V-A* law; Coulomb corrections are included and nuclear matrix elements occur as parameters that can frequently be determined from terrestrial experiments. There is no atomic binding-energy contribution to the total beta-decay energy for completely ionized atoms and this causes a decrease in decay rates for low-energy electron emitters in stars relative to their terrestrial values. Screening will usually not affect beta-decay rates significantly. The exclusion principle inhibits beta decay in stellar interiors because many of the low-momentum states are occupied prior to the decay; the amount by which a decay rate is decreased can be calculated in terms of the known beta spectrum and the temperature and density of the medium surrounding the radioactive nucleus. Beta decay for some normally radioactive nuclei is almost impossible in the interior of very dense stars, such as white dwarfs, since the Fermi energy can equal or exceed the maximum beta-decay energy available. Some applications to the theory of element formation in stars are suggested.

I. INTRODUCTION

We discuss four effects that cause beta-decay half-lives in stellar interiors to differ from their terrestrial values. Our results are useful for studies of heavy element production in stellar interiors, since accurate calculations of element abundances must take account of variations in beta-decay half-lives. Moreover, changes in the decay rates depend upon the temperature and density of stellar matter and therefore vary with stellar class and with position in a given star. This variation of decay rates with stellar class may help to explain some of the anomalous abundances that have been observed.[1] The temperature and density dependence of beta-decay rates must also be considered in discussing equilibrium configurations of stars.

The possibility that electron emission rates in stars differ significantly from their values on earth was apparently first suggested by Daudel and his associates[2] in 1947. Daudel *et al.* suggested that a nucleus could beta decay by creating an electron in a bound atomic orbit instead of in a continuum state and pointed out that this process would be most important in stars. Allowed bound-state beta decay was subsequently discussed by several authors[3] and the subject has recently been re-examined by the present author.[4] The possible significance of bound-state beta decay for the theory of atomic abundances was pointed out by the present author[4] and several nuclei were listed which, if completely ionized, are more likely to decay by creating an electron in a bound instead of a continuum state. In connection with this work on bound-state decay in stars, Cox and Eilers[5] of the Los Alamos Scientific Laboratory have computed the probability of finding an electron in the *K* shell of several heavy elements at temperatures of 10^{+8} and 10^{+9} °K and densities of 10^{+2} and 10^{+4} g/cm^3. The preliminary results of Cox and Eilers are shown in Tables I and II and indicate a high degree of ionization of heavy elements in stellar interiors. These results are important in our discussion of electron capture rates; they also imply that internal conversion coefficients are decreased in stellar interiors.

There is a small probability that excited states of nuclei in stellar interiors are occupied and these excited states can undergo beta decay with half-lives that are much different from the terrestrially measured half-lives of the ground states. Cameron[6] has discussed the

TABLE I.[a] Probable occupation of the *K* shell for a density of 10^{+2} g/cm^3 at 10^{+8} and 10^{+9} °K. A heavy element abundance of 1×10^{-6} was assumed with the remainder He.

Atomic number	Element	Occupation at 10^{+8} °K	Occupation at 10^{+9} °K
6	C	0.0126	0.0004
14	Si	0.0156	0.0004
28	Ni	0.0374	0.0004
41	Nb	0.140	0.0005
44	Ru	0.200	0.0005
46	Pd	0.256	0.0005
47	Ag	0.290	0.0005
62	Sm	1.33	0.0007
63	Eu	1.42	0.0007
76	Os	1.95	0.0010

[a] Los Alamos Scientific Laboratory data obtained from A. N. Cox and D. D. Eilers.

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[1] See, for example, E. M. Burbidge, G. R. Burbidge, W. A. Fowler, and F. Hoyle, Revs. Modern Phys. **29**, 547 (1957); A. G. W. Cameron, Ann. Rev. Nuclear Sci. **8**, 299 (1958); A. G. W. Cameron, Atomic Energy of Canada Limited Report CRL 41, 1957 (unpublished), 2nd ed.

[2] R. Daudel, M. Jean, and M. Lecoin, J. phys. radium **8**, 238 (1947); Compt. rend. **225**, 290 (1948); R. Daudel, P. Benoist, R. Jacques, and M. Jean, *ibid.* **224**, 1427 (1947).

[3] P. M. Sherk, Phys. Rev. **75**, 789 (1949); E. Galzenati, M. Marianaro, and S. Okubu, Nuovo cimento **15**, 934 (1960).

[4] J. N. Bahcall, Phys. Rev. **124**, 495 (1961).

[5] The results in Tables I and II have been obtained privately from Dr. A. N. Cox and Dr. D. D. Eilers of Los Alamos Scientific Laboratory. I am grateful to Dr. Cox for permission to quote these results and for informative communications concerning the method by which they were obtained.

[6] A. G. W. Cameron, Astrophys. J. **130**, 452 (1959).

astrophysical importance of such decays, which he calls photo-beta reactions.

In this paper, we consider the effect on beta-decay rates of electron capture from continuum orbits, the absence of atomic binding energies, screening, and the exclusion principle. We calculate the rate of allowed electron capture from continuum orbits in a Fermi gas using the $V-A$ law and derive equations from which many electron capture rates in stars can be computed from their terrestrially observed rates. These equations are independent of nuclear matrix elements but depend upon the temperature and density of the stellar matter. The absence, for completely ionized atoms, of the atomic binding energy contribution to the total decay energy causes a decrease in decay rates for very-low-energy electron emitters. Screening effects in beta decay are usually too small to be of astrophysical importance. The exclusion principle inhibits beta decay in stellar interiors because many of the low-momentum states are occupied prior to the decay. We derive equations from which the decrease in beta-decay rates due to the exclusion principle can be calculated if the temperature and density of the stellar matter are known; these equations are independent of nuclear matrix elements for many interesting cases. We list a large number of nuclei whose decay rates in dense stellar interiors can be significantly decreased by the exclusion principle.

II. ELECTRON CAPTURE

a. Capture of an Individual Electron

The capture of electrons from continuum orbits in stars has been investigated by Dumezil-Curien and Schatzman[7]; they studied the reactions

$$H^1+e^- \rightarrow n+\nu, \tag{1a}$$

$$He^3+e^- \rightarrow H^3+\nu, \tag{1b}$$

$$Be^7+e^- \rightarrow Li^7+\nu, \tag{1c}$$

as a function of temperature and density. Reactions (1b) and (1c) are especially interesting since at high densities they can occur in the modified proton-proton and helium-helium reactions, respectively.[8,9]

The calculations of Dumezil-Curien and Schatzman[7] were performed using the original Fermi (pure vector) form of the β-interaction and plane waves for the state vectors of the captured electrons. Nuclear matrix elements were set equal to one. In this section, we reinvestigate electron capture from continuum orbits using the now established $V-A$ interaction; nuclear matrix elements are presented explicitly since they can often be determined from terrestrial experiments. We shall find that the nuclear Coulomb field, which modifies the wave function of an incident electron, has a large influence on the magnitude of the capture rates and on their temperature dependence.

We make the usual assumptions of allowed beta decay: (1) Nuclei are treated nonrelativistically; (2) the lepton current is evaluated at the nuclear surface. The beta-decay interaction is

$$H_\beta = G2^{-\frac{1}{2}}[\bar{\psi}_\nu\gamma_\alpha(1+\gamma_5)\psi_e][\bar{\psi}_n\gamma_\alpha(C_V-C_A\gamma_5)\psi_p] + \text{H.c.}, \tag{2}$$

where all symbols have their usual meaning.[10] We calculate the rate for electron capture from a continuum orbit by a nucleus of arbitrary charge Z. In calculating this rate, it is necessary to use for the initial state a Coulomb distorted plane wave with an outgoing spherical wave; in ordinary beta decay, a Coulomb distorted plane wave with an ingoing spherical wave is required for the final state.[11] It is shown in the Appendix that the value of $|\psi_e|_{\text{nucleus}}^2$ is the same for both cases.

The allowed capture rate for a single nucleus of charge Z, nuclear energy release, W_0, for one electron with total energy W in a volume V is[12]

$$\lambda = G^2(W_0+W)^2[C_V^2\langle 1\rangle^2 + C_A^2\langle\sigma\rangle^2]F(Z,W)/2\pi\hbar^4c^3V. \tag{3}$$

Equation (3) yields the result of Dumezil-Curien and Schatzman if we set $C_A=0$, $F(Z,W)\cong F(0,W)=1$, and

TABLE II.[a] Probable occupation of the K shell for a density of 10^{+4} g/cm^3 at 10^{+8} and 10^{+9} °K. A heavy element abundance of 1×10^{-6} was assumed with the remainder He.

Atomic number	Element	Occupation at 10^{+8} °K	Occupation at 10^{+9} °K
6	C	0.00	0.0000
14	Si	0.00	0.0000
28	Ni	1.21	0.0416
41	Nb	1.67	0.0470
44	Ru	1.75	0.0488
46	Pd	1.80	0.0500
47	Ag	1.82	0.0507
62	Sm	1.98	0.0641
63	Eu	1.98	0.0653
76	Os	2.00	0.0848

[a] Los Alamos Scientific Laboratory data obtained from A. N. Cox and D. D. Eilers.

[7] P. Dumezil-Curien and E. Schatzman, Ann. astrophys. **13**, 80 (1950); P. Dumezil-Curien, *ibid.* **14**, 40 (1951); P. Dumezil-Curien and E. Schatzman, *ibid.* **14**, 46 (1951); and E. Schatzman, *ibid.* **16**, 162 (1953). Electron capture in stellar interiors had been previously considered by H. A. Bethe, Phys. Rev. **55**, 434 (1939) and C. L. Critchfield, Astrophys. J. **96**, 1 (1942).

[8] E. Schatzman, *White Dwarfs* (Interscience Publishers, Inc., New York, 1958), Chap. 6. The results in this chapter should be modified by including the effects of the nuclear Coulomb field and the empirical evaluation of nuclear matrix elements. These effects are most significant for reaction (1c).

[9] The rate for reaction (1c) has been estimated by A. G. W. Cameron, Atomic Energy of Canada Limited Report CRL 41, 1957 (unpublished), 2nd ed. Cameron replaced, in the usual formula for electron capture, the bound electron wave function at the nucleus by an approximate form of the continuum electron wave function at the nucleus [our Eq. (7)].

[10] E. J. Konopinski, Ann. Rev. of Nuclear Sci. **9**, 99 (1959).

[11] See, for example, G. Breit and H. A. Bethe, Phys. Rev. **93**, 888 (1954).

[12] We use a definition of reduced matrix elements suggested by E. J. Konopinski

$$\langle I'(M')|S_{Jm}^{\dagger}|I(M)\rangle = \langle I'(M')J(m)|I(M)\rangle\langle S_J\rangle.$$

This definition simplifies a number of formulas in the theory of beta decay.

$\langle 1\rangle=1$. The function $F(Z,W)$ is the well-known ratio of the electron density at the nucleus calculated with a Coulomb distorted plane wave to the density calculated with a plane wave. If $\alpha^2Z^2\ll 1$ and the electron velocity is not relativistic,[13] then

$$F(Z,W)\cong 2\pi\eta/(1-e^{-2\pi\eta}), \quad (4)$$

where

$$\eta=(\alpha Z)c/v, \quad (5)$$

and v is the velocity of the electron that is captured. Thus F, and hence λ, depends on the temperature T through the quantity

$$\eta\cong\alpha Z(mc^2/3kT)^{\frac{1}{2}}, \quad (6)$$

where we have inserted for v its average value according to the equipartition theorem. F is a monotonically decreasing function of Z and v^{-1}.

If $2\pi\eta\gg 1$, then

$$F(Z,W)\cong\alpha Z(2\pi)(mc^2/3kT)^{\frac{1}{2}}. \quad (7)$$

For $Z=4$, approximation (7) is valid for temperatures $\lesssim 10^7$ °K. For larger Z, this approximation is valid for somewhat higher temperatures. If α^2Z^2 is not small or very high electron velocities are considered, the relativistic formula for $F(Z,W)$ must be used; this formula is given in the Appendix. If the nuclei are not completely ionized, $F(Z,W)$ must be modified slightly to take account of electron screening.

Equation (3) corresponds to a cross section for electron capture that is

$$\sigma=2V\lambda/v. \quad (8)$$

A typical cross section computed from Eq. (8) is of the order of 10^{-43} or 10^{-44} cm^2, for $W_0=mc^2$. This cross section is of the same order of magnitude as the neutrino capture cross section, 3×10^{-43} cm^2, measured by Reines and Cowan.[14]

Equations (3) and (8) are valid for any W satisfying

$$W_0+W\geq 0.$$

If $W_0<-mc^2$, as in reactions (1a) and (1b), Eq. (8) gives the cross section for induced electron catpure.

The usual allowed selection rules are apparent in Eq. (3); they are a nuclear spin change of 0 or 1 and no parity change. If a nucleus decays terrestrially by allowed electron or positron emission or by allowed electron capture, the only unknown quantities in Eq. (3), W_0 and $[C_V^2\langle 1\rangle^2+C_A^2\langle\sigma\rangle^2]$, can be determined experimentally. If the nucleus (Z,A) decays terrestrially by allowed electron capture, the allowed approximation to the general result of Konopinski[13] for electron capture from bound orbits leads to the relation

$$\frac{\lambda_{\text{star}}(W)}{\lambda_{\text{earth}}}\cong\frac{(W_0+W)^2}{2(W_0-b)^2}\frac{|\psi_e|_{\text{star}}^2}{|\psi_e|_{\text{earth}}^2}, \quad (9)$$

where b is the binding energy of the electron captured in the terrestrial decay and $|\psi_e|_{\text{earth}}^2$ is the density of bound electrons at the nuclear surface. The factor of two arises from the sum over both spin directions in the calculation of λ_{earth}; the quantity λ_{star} was calculated by averaging over spins. The ratio in Eq. (9) is independent of nuclear matrix elements. A similar relation without the energy dependent factors, has been used by Cameron to study photobeta reactions in stellar interiors.[15]

In the usual notation,[16]

$$|\psi_e|^2_{\text{earth}}\cong(4\pi)^{-1}g_{1,-1}^2. \quad (10)$$

Accurate values for $|\psi_e|_{\text{earth}}^2$ have been computed by Brysk and Rose[17]; these authors include effects due to electron screening, finite nuclear size, and an average of electron density over nuclear volume. A crude approximation that is sufficient for many purposes is

$$g_{1,-1}^2\cong 4Z^3/a_0^3. \quad (11)$$

This approximation greatly underestimates g^2 for heavy nuclei (by about a factor of 7 for $Z=90$) but gives a fair approximation for $Z<40$.[13]

b. Capture in a Fermi Gas

Formulas (3) and (8) are valid for an electron with a fixed energy W. We now assume that the electrons in a stellar interior constitute a perfect Fermi gas; Schatzman[18] has reviewed the arguments favoring this assumption. We note in addition that the average electrostatic interaction energy is of the order of $(4n_e)^{\frac{1}{3}}e^2$ (where n_e is the number of electrons per unit volume); this energy is usually less than 1 kev, while the average kinetic energy is of the order of 10 or 100 kev. This fact suggests that the Coulomb interactions will perturb the location of single-particle levels but will not change greatly their location or relative probability of being occupied.

The electron capture rate in a perfect Fermi gas is

$$\lambda=\frac{G^2\zeta K}{2\pi^3(\hbar/mc)^7m^2c^3}, \quad (12)$$

where ζ represents the nuclear matrix element combination

$$\zeta=[C_V^2\langle 1\rangle^2+C_A^2\langle\sigma\rangle^2], \quad (13)$$

[13] E. J. Konopinski (manuscript on the theory of beta decay to be published by Oxford University Press).

[14] F. Reines and C. Cowan, Phys. Rev. **113**, 272 (1959).

[15] A. G. W. Cameron, Astrophys. J. **130**, 452 (1959). See also reference 9. A relation of the same form as that given by Cameron has been used independently by W. A. Fowler (private communication).

[16] M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961).

[17] H. Brysk and M. E. Rose, Revs. Modern Phys. **30**, 1169 (1958).

[18] Reference 8, Chapter 4.

and the dimensionless quantity K is given by[19]

$$K=m^{-5}c^{-7}\int_0^\infty dp\, p^2 \frac{F(Z,W)(W_0+W)^2}{\exp(-\nu+W/kT)+1}. \quad (14)$$

This result follows from an integration of Eq. (3) over electron momenta using Fermi-Dirac statistics. The quantity $e^{-\nu}$ can be determined from the temperature and density of the electron gas. For a degenerate Fermi gas, $e^{-\nu}$ is defined to be zero; its value for Boltzmann statistics is given in Eq. (16) below. The definition in Eq. (14) for K assumes that $W_0>-mc^2$. If $W_0<-mc^2$, the lower limit in Eq. (14) must be replaced by

$$P_0\equiv c^{-1}(W_0^2-m^2c^4)^{\frac{1}{2}}.$$

Any departures from a perfect Fermi gas can be readily incorporated into Eq. (14) by modifying appropriately the statistical factor occurring in the integrand.

i. Boltzmann Statistics

If Boltzmann statistics apply, we can write

$$K_B\cong m^{-5}c^{-7}e^{+\nu}\int_0^\infty dp\, p^2 e^{-W/kT}F(Z,W)(W_0+W)^2, \quad (15)$$

where

$$\exp(-\nu+mc^2/kT)\cong\frac{2}{n_e}\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\left(1+\frac{15}{8}\frac{kT}{mc^2}\right), \quad (16)$$

and n_e is the number of electrons per unit of volume.

If $2\pi\eta_{\rm av}\ll 1$, $\alpha^2Z^2\ll 1$, and $kT\ll mc^2$, the integral in Eq. (15) can be evaluated by the method of steepest descent and[20]

$$K_B\cong\pi\alpha Zn_e(2\pi)^{\frac{3}{2}}(kT)^{-\frac{1}{2}}W_0^2\left(1-\frac{15}{8}kT\right)$$
$$\times\Bigg[(1+2\mu_0+2\mu_0^2)+\frac{2^{\frac{1}{4}}(1+x_m/W_0)^2x_m^{\frac{5}{4}}e^{-f(x_m)}}{(3\alpha Zk^2T^2)^{\frac{1}{2}}}$$
$$\left\{1+\mathrm{erf}\left(\left[\frac{f^{11}(x_m)x_m^2}{2}\right]^{\frac{1}{2}}\right)\right\}\Bigg], \quad (17a)$$

$$x_m=2^{-\frac{1}{3}}[\pi\alpha ZkT]^{\frac{2}{3}}, \quad (17b)$$

$$f(x)=(x/kT)+\pi\alpha Z(2/x)^{\frac{1}{2}}, \quad (17c)$$

where $\mu_0=kT/W_0$. A crude approximation to Eqs. (12) and (17) can be obtained by substituting $W=\frac{3}{2}kT$ in Eq. (3) and multiplying the result by the average number of electrons in a volume V. This approximation underestimates the capture probability.

If $2\pi\eta_{\rm av}\ll 1$, $\alpha^2Z^2\ll 1$, and $kT\ll mc^2$, then[21]

$$K_B\cong\pi^2W_0^2n_e. \quad (18)$$

ii. Degenerate Statistics

Electron capture from continuum orbits is most important for extremely high densities such as may be found in white dwarfs. To a good approximation, the electrons in these very dense stars can be treated as a degenerate Fermi gas.[22] The appropriate K is

$$K_D=\int_0^{P_F} dp\, p^2F(Z,W)(W_0+W)^2, \quad (19)$$

where

$$P_F=(3\pi^2n_e)^{\frac{1}{3}}, \quad (20)$$

is the Fermi momentum. The Fermi energy, E_F, can be comparable to or larger than W_0 in very dense stars.

If $2\pi\eta_F\gg 1$, and $\alpha^2Z^2\ll 1$,

$$K_D=\frac{2\pi\alpha Z}{5}\left[\left(E_F^5+\frac{5}{2}W_0E_F^4+\frac{5}{3}W_0^2E_F^3\right)-\left(1+\frac{5}{2}W_0+\frac{5}{3}W_0^2\right)\right], \quad (21)$$

and if $2\pi\eta_F\ll 1$,

$$K_D=\tfrac{1}{5}P_F^5+\tfrac{1}{3}P_F^3(W_0^2+1)+\tfrac{1}{4}W_0[2P_FE_F^3-P_FE_F-\ln(P_F+E_F)]. \quad (22)$$

c. Astrophysical Applications

The total rate for allowed electron capture in a star can be expressed in terms of the terrestrially measured rate for allowed electron capture by the relation

$$\lambda_{\rm star}/\lambda_{\rm earth}=N_K+R, \quad (23)$$

where N_K is the probable number of electrons in the K shell of the stellar atom and R is the ratio of the capture rate from a continuum orbit to the capture rate from a bound orbit. The ratio R can be computed from Eqs. (3), (9), (12), and (14); an order-of-magnitude estimate can be obtained from Eq. (12) alone:

$$R\cong\frac{n_eF(Z,\frac{3}{2}kT)}{(2/\pi)(Z/a_0)^3}. \quad (24)$$

For $n_e=10^{28}/\mathrm{cm}^3$ and $T=10^8$ °K,

$$R\cong 200Z^{-2}, \quad (25)$$

for $Z\gtrsim 20$. Equations (3) and (24), in combination with Tables I and II show that electron capture rates in stars can be orders of magnitude smaller than terrestrially measured capture rates. Moreover, electron capture rates depend sensitively on temperature and

[19] We use the notation of D. ter Haar, *Elements of Statistical Mechanics* (Rinehard and Company, Inc., New York, 1954). Appendix VI of this book describes relativistic statistics and is particularly relevant to our discussion.

[20] We set $\hbar=m=c=1$ in the rest of this section.

[21] We also assume that $W_0\ll kT$.

[22] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover Publications, Inc., New York, 1957), Chap. XI.

density, since both N_K and R are sensitive functions of temperature and density. Electron capture may occur with only a small probability in the interior of giant stars and supernovae, while even induced electron capture may be of some importance in white dwarfs.

Terrestrially measured electron capture rates have been used for a number of purposes in the literature on nucleogenesis.[23] For example, the half-life of Co^{57} has been used to discuss the equilibrium associated with the iron peak in the atomic abundance curve; the half-life of Eu^{152} has been used as an indicator of the s-process time scale; and the half-life of Ti^{44} has been used to estimate the time scale for the α-process. These considerations must be revised if further calculations on the ionization of heavy elements in stellar interiors definitely establish that even the heaviest elements are completely ionized.[24]

III. ATOMIC BINDING ENERGIES

The total energy available for beta decay on earth is the sum of the nuclear-energy release, W_0, and the difference between initial and final atomic-binding energies. If $W_0 < mc^2$, the terrestrial beta decay is possible only because the absolute value of the electron binding energies increases when the nuclear charge increases from Z to $Z+1$. The quantity $W_0 - mc^2$ is probably less than[25] -5 kev for the isotope Re^{187} and hence Re^{187} cannot decay in a stellar interior if the temperature is high enough for the nucleus to be completely stripped of atomic electrons. We assume that the electrostatic energy of the free particles surrounding the Re^{187} nucleus would not decrease by 5 kev or more if the nuclear charge were increased by one unit; crude estimates indicate that this assumption is valid for densities less than 10^4 g/cm^3. For Pu^{241} and the 85% branch of Pb^{210}, $W_0 - mc^2$ may also be negative.[26] In order to determine definitely the sign of $W_0 - mc^2$ for the latter two nuclei, more accurate measurements of the maximum electron kinetic energy in the terrestrial decays are necessary and a more accurate theoretical estimate of the change in atomic binding energies is also required.[27] The decay of Pb^{210} and Pu^{241} is at least greatly inhibited if they are completely ionized, since a large part of the terrestrially available phase space is inaccessible to electrons emitted by the stripped nuclei.

The decrease in the decay rate of an electron emitting nucleus due to its ionization can easily be calculated. The change in the decay rate is

$$\Delta\lambda = \int_{W_0}^{W_0+\Delta b} \lambda(W) dW, \qquad (26)$$

where $\lambda(W)$ is the beta spectrum function and Δb is the absolute value of the change in atomic binding energy when the charge of the terrestrial atom changes from Z to $Z+1$. The function $\lambda(W)$ is known theoretically for all degrees of forbiddenness. The ratio $\Delta\lambda/\lambda^0$ of the change in the decay rate to the unperturbed rate is independent of nuclear matrix elements for allowed, first-forbidden nonunique, and all unique decays. This independence of nuclear matrix elements exists because, for the cases mentioned, $\lambda(W)$ contains the nuclear matrix elements only as an energy independent factor. The ratio $\Delta\lambda/\lambda^0$ is negligible if $\Delta b/(W_0 - mc^2) \ll 1$.

IV. SCREENING

The main screening effect of the free electrons, protons, alpha particles, and other ions surrounding a heavy nucleus can be approximated by adding a spherically symmetric potential to the Dirac equation that describes the electron created in the beta decay process.[28] The screening potential will change the radial dependence of the wave function for the created electron and hence will alter the beta-decay rate.

Reitz[29] has treated the effect of electron screening on terrestrial beta-decay rates by adding a spherically symmetric potential, derived mainly from the Thomas-Fermi model of the atom, to the Dirac equation. He found that screening changed the total decay rates by at most 10 to 15% and was negligible for large decay energies. The results of Reitz show that positron-emission rates are increased and electron-emission rates are decreased by screening in a terrestrial atom. He also found that screening is more important for positron decays than for electron decays.

The ratio, R, of the free electron density at the nucleus in a star compared to the bound-electron density at the nucleus for a terrestrial atom has been previously considered in Eq. (24). Except for the densest stars, the quantity R is less than 1 and this

[23] E. M. Burbidge, G. R. Burbidge, W. A. Fowler, and F. Hoyle, Revs. Modern Phys. **29**, 547 (1957).

[24] I am grateful to Professor W. A. Fowler for describing (private communication) how the conclusions expressed in reference 23 must be modified if electron capture does occur with only a negligible probability.

[25] The measurement of the maximum electron kinetic energy has been performed by A. D. Suttle, Jr., and W. F. Libby, Phys. Rev. **95**, 866 (1954). The change in atomic binding energies can be estimated from the work of L. Foldy, *ibid.* **83**, 397 (1951); see also reference 26.

[26] The fact that a completely ionized Pu^{241} nucleus might not be able to beta decay was first pointed out by M. S. Freedman, F. Wagner, Jr., and D. W. Engelkemeir, Phys. Rev. **88**, 1155 (1951). These authors performed the most accurate measurement of the electron end point energy in Pu^{241} decay.

[27] The experimental measurements of the maximum electron kinetic energy and the theoretical estimates of the change in atomic binding energies are both subject at present to appreciable uncertainties. The results of Foldy (reference 25) on electron binding energies are based upon a number of Hartree-Fock calculations, many of which do not include relativistic effects. Hence, Foldy's results are not very accurate when applied to an atom of large atomic number.

[28] A clear physical picture of the average arrangement of electrons and ions surrounding a heavy nucleus has been given by G. Keller and R. E. Meyerott, Argonne National Laboratory Report—4771, 1952 (unpublished).

[29] J. R. Reitz, Phys. Rev. **77**, 10 (1950). See also M. E. Rose, *ibid.* **49**, 727 (1936) and other references cited in Reitz's paper.

indicates that screening effects are usually smaller in stellar interiors than on earth.[30] Thus, positron rates will usually be decreased in stars and electron rates will usually be increased, if the only change considered is due to screening. The magnitude of the correction can be estimated from the work of Reitz[29] and Rose,[29] but it will usually be unimportant for astrophysical problems.

V. THE EXCLUSION PRINCIPLE

The total decay rate for electron emission from unoriented nuclei is usually expressed as an integral of the form

$$\lambda_0 = 2\int_0^{P_0} dn\, \lambda(p), \tag{27}$$

where $2dn$ is the number of electron states with momentum between p and $p+dp$, $\lambda(p)$ is the decay probability per unit of time for a fixed momentum p, and

$$P_0 = c^{-1}(W_0^2 - m^2c^4)^{\frac{1}{2}}. \tag{28}$$

The free-electron concentrations that exist in stellar interiors are sometimes large enough that the number of states available to the decay electron is significantly decreased in accordance with the exclusion principle.[31] The effect of the exclusion principle may be taken into account by replacing dn in Eq. (27) by

$$dn(1-S) = (p^2dp/2\pi^2)(1-S), \tag{29}$$

where S is the probability that the free-electron state with momentum p is occupied. Assuming the beta decay takes place in the presence of a perfect Fermi gas of electrons,

$$\lambda = \pi^{-2}\int_0^{P_0} dp\, p^2\{1-[\exp(-\nu+W/kT)+1]^{-1}\} \tag{30}$$

is the appropriate generalization of Eq. (27). For accurate calculations pertaining to very dense stars, it may be necessary to take account of the departure of S from the value appropriate to a perfect Fermi gas[18]; this possibility has been discussed briefly in Sec. II(b). It is convenient to consider separately the change, $\Delta\lambda$, in the decay rate, where

$$\Delta\lambda = \lambda_0 - \lambda$$
$$= \pi^{-2}\int_0^{P_0} dp\, p^2\lambda(p)[\exp(-\nu+W/kT)+1]^{-1}. \tag{31}$$

[30] This assumes that the temperature is high enough that the decaying nucleus is largely stripped of bound electrons.

[31] A. G. W. Cameron, reference 9, page 123, pointed out that the 2.8-Mev decay of Mn^{56} is slowed down at high densities because there is no phase space available into which low-energy electrons can be emitted. Cameron concluded that the effect of the exclusion principle is less important than the occupation of excited states of Mn^{56}, for the case considered.

Since the theoretical spectrum $\lambda(p)$ is known for beta transitions of all degrees of forbiddenness, λ or $\Delta\lambda$ can in principle be calculated for all cases of interest. However, even λ_0 cannot be evaluated analytically for the simplest case of allowed decays except with simplifying assumptions. The difficulty in performing the integration analytically is due to the appearance in the allowed spectrum of the complicated Coulomb density ratio, $F(Z,W)$. We shall illustrate the effect of the exclusion principle by considering some simple but important special cases.

a. Boltzmann Statistics

Boltzmann statistics is valid if

$$\exp[\nu-(kT)^{-1}] \ll 1. \tag{32}$$

If the inequality (32) is satisfied, the electron concentration is not large enough to change the beta-decay rate significantly, since it follows from Eqs. (30), (31), and (32) that

$$\Delta\lambda_B/\lambda_0 < \exp[\nu-(kT)^{-1}] \ll 1. \tag{33}$$

b. Degenerate Statistics

For a degenerate Fermi gas,

$$\Delta\lambda_D/\lambda_0 = \int_0^{P_F} dp\, p^2\lambda(p) \Big/ \int_0^{P_0} dp\, p^2\lambda(p). \tag{34}$$

If $P_0 \leq P_F$, then

$$\Delta\lambda_D/\lambda_0 = 1 \tag{35}$$

and no beta decay is possible. This corresponds physically to the prior occupation of all states into which the decay electron could be emitted without violating conservation of energy.

For all cases of allowed and first-forbidden nonunique decays,

$$\lambda(p) \propto (W_0-W)^2F(Z,W). \tag{36}$$

If $2\pi\eta_F \gg 1$ and $\alpha^2Z^2 \ll 1$, then Eqs. (34) and (36) lead to the following result for all cases in which the nuclear spin changes by 0 or 1 unit:

$$\Delta\lambda_D/\lambda_0 = D_1(E_F,W_0)/D_1(W_0,W_0), \tag{37}$$

where

$$D_1(x,W_0) \equiv x^5 - (5/2)W_0x^4 + (5/3)W_0^2x^3$$
$$- (1 - 5_0W/2 + 5W_0^2/3). \tag{38}$$

If $2\pi\eta_F \ll 1$ and $\alpha^2Z^2 \ll 1$, then for all cases in which the nuclear spin changes by 0 or 1 unit

$$\Delta\lambda_D/\lambda_0 = D_2(P_F,W_0)/D_2(P_0,W_0), \tag{39}$$

where

$$D_2(x,W_0) = (x^5/5) + (W_0^2+1)x^3/3$$
$$- (W_0/4)\{2x(1+x^2)^{\frac{3}{2}} - x(1+x^2)^{\frac{1}{2}}$$
$$- x\ln[x+(1+x^2)^{\frac{1}{2}}]\}. \tag{40}$$

c. Astrophysical Applications

Equations (37) and (39) have an important feature in common: the ratio $\Delta\lambda/\lambda^0$ is independent of nuclear matrix elements. This feature is a general property of allowed, first-forbidden nonunique, and all unique transitions; it exists because for the cases mentioned $\lambda(p)$ involves nuclear matrix elements only as an energy independent factor. Thus one can calculate, for the cases mentioned, stellar decay rates from the terrestrial rates, if the temperature and density of the electron gas are known.

The electrons in the interior of white dwarf stars are well described by the degenerate Fermi statistics.[22] An electron concentration of $n_e = 7\times 10^{+29}$ cm^{-3}, which is a reasonable estimate of the concentration in some white dwarfs,[22] corresponds to a Fermi energy of 200 kev. The following nuclei decay on earth by emitting electrons with maximum kinetic energy ≤ 200 kev: H^3, Si^{32}, S^{35}, Ni^{63}, Ni^{66}, Se^{79}, Zr^{93}, Nb^{95}, Pd^{107}, I^{129}, Sm^{151}, Tm^{171}, Re^{187}, Os^{191}, Hg^{203}, Pb^{210}, Ac^{227}, Ra^{228}, Pu^{241}, Pu^{246}, and Bk^{249}.

The nuclei listed above may not be able to beta decay inside white dwarf stars. Moreover, many additional nuclei have maximum beta decay energies not much greater than 200 kev and would have their beta decay rates greatly reduced in a Fermi sea of electrons with $E_F \approx 200$ kev. The exclusion principle may also be important in inhibiting beta decay in the interior of dense red giant stars.

VI. CONCLUSION

We have investigated four effects that cause beta-decay half-lives to vary with temperature and density and have shown that the difference between terrestrial and stellar decay rates is large for many nuclei. If the temperature and density of the stellar matter are known, the altered decay rates can often be calculated from their terrestrially measured values.

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APPENDIX

The appropriate wave function for electron capture from continuum orbits is, in the usual notation,[16]

$$\psi_\rho = \sum_{\kappa,\mu} i^{+l}\langle l(\mu-\rho)\tfrac{1}{2}(\rho)\,|\,j(\mu)\rangle Y_{l,\mu-\rho}{}^*(\hat{p})e^{+i\delta_\kappa}\psi_{\kappa,\mu}, \quad \text{(A1)}$$

where ψ_ρ is asymptotically an incoming plane wave plus an outgoing spherical wave, $\psi_{\kappa,\mu}$ is a Coulomb spherical wave, and δ_κ is the difference between the Coulomb phase shift exclusive of the logarithmic term and the $Z=0$ phase shift. The index ρ denotes the spin projection of the incident wave. For beta emission, ψ_ρ includes an incoming spherical wave, and δ_κ in Eq. (A2) must be replaced by $-\delta_\kappa$.

Evaluating ψ_ρ at the nuclear surface and omitting parity-changing terms proportional to $\boldsymbol{\alpha}\cdot\mathbf{r}$, we find after some calculation that

$$\psi_\rho(r\to 0) = e^{+i\delta_{-1}}\left(\frac{g_{-1}}{4\pi}\right)\begin{pmatrix}\chi_\rho \\ (f_{+1}/g_{-1})e^{+i(\delta_{+1}-\delta_{-1})}\boldsymbol{\sigma}\cdot\mathbf{p}\chi_\rho\end{pmatrix}. \quad \text{(A2)}$$

where χ_ρ is a two-component Pauli spinor. The form of the $\boldsymbol{\alpha}\cdot\mathbf{r}$ terms and the reason for omitting them is discussed briefly in the Appendix of reference 4. It is easy to show from Eq. (A2) that $\psi_\rho{}^\dagger(1+\gamma_5)\psi_\rho$ depends only on the cosine of the difference $\delta_{+1}-\delta_{-1}$ and hence is unchanged by the transformation $\delta_\kappa\to-\delta_\kappa$. Thus the electron density at the nucleus is the same for allowed electron capture from continuum orbits and allowed electron emission, as asserted in the main text. Using the known forms of f and g, the relativistic expression for $F(Z,W)$ can be calculated from Eq. (A2); it is[13]:

$$F(Z,W) = 4\left(\frac{2pR}{\hbar}\right)^{-2(1-\gamma)} e^{+\pi\eta}\frac{|\Gamma(\gamma+i\eta)|^2}{[\Gamma(2\gamma+1)]^2}, \quad \gamma=(1-\alpha^2Z^2)^{\frac{1}{2}}, \quad \eta=\alpha ZW/cp. \quad \text{(A3)}$$

The relativistic definition of η has been used in Eqs. (21) and (38) of the text.