Alternative electrostatic Green's function for a long tube

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This note describes an expression for the electrostatic Green's function in a long conducting tube. The expression allows one to readily compute the potentials and fields at and in the vicinity of the singularity where other methods have difficulty.
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In recent years several authors and groups have tackled the problem of computing the force on a charged particle due to the charge it induces on the surface of nearby conductors.[1–5] Although solutions for simple boundary conditions are well known formally,[6,7] the representations are generally unsuited for evaluation at or near the source. By building on some of this recent work, an alternative representation of the electrostatic Green's function for an infinitely long tube can be found in the form:

$$\Phi(x;x')=\frac{q}{4\pi\epsilon_0}\left(\frac{1}{|x-x'|}-H(x;x')\right), \tag{1}$$

where x' is the location of a point charge, the source. The function $H(x;x')$ is a solution of the Laplace equation and is determined by the boundary conditions. Most of the known solutions to Eq. (1) involve multiple sums over products of transcendental functions and incorporate both terms on the right-hand side in such a way that the boundary condition $\Phi(x=\mathrm{wall};x')=0$ is enforced with each individual term of the sum. This is a "convienence" not a necessity.

The alternative here is to find an explicit expression for $H(x;x')$ to which the free space Coulomb potential can be added. Since $H(x;x')$ contains no poles, it is behaves well throughout the volume of interest.

Note for simplicity of notation, $q/4\pi\epsilon_0$ is set equal to unity and all distances are scaled to the cylinder radius, r_0.

The evaluation of $H(x;x')$ begins by replacing the real cylinder wall with a "virtual" cylinder and the potential everywhere on this cylindrical boundary is found from Coulomb's law. If the center of the cylinder is located at the origin and the point charge in cylindrical coordinates (r,θ,z) is placed at $(r',0,0)$ then

$$\Phi(r_0,\theta,z;r',0,0)=\frac{1}{\sqrt{1+z^2+r'^2-2r'\cos(\theta)}}. \tag{2}$$

$\Phi(r_0,\theta,z;r',0,0)$ is illustrated in Fig. 1 by a contour plot on a cylindrical surface. A solution to $H(x;x')$ can be constructed with the general form,

$$H(x;x')=a\sum_{m=-\infty}^{\infty}\cos(m\theta)\times\int_0^{\infty}A_m(k,r')I_m(kr)\cos(kz)\,\mathrm{d}k. \tag{3}$$

Here, $I_m(x)$ is the Bessel function of the first kind with imaginary argument and a is a normalization constant. The challenge is then to find $A_m(k,r')$ subject to the boundary condition, Eq. (2). Hess and Chen[1] pointed out two results from work by Watson[8] that are useful. The first is an integral representation of the Bessel function of the second kind of imaginary argument [Watson's Eq. (1), Sec. 6.16]:

$$K_0(xz)=\int_0^{\infty}\frac{\cos(kx)}{\sqrt{k^2+z^2}}\,\mathrm{d}k. \tag{4}$$

The second result follows from Watson's Eq. (8), Sec. 11.3,

$$\chi=\sqrt{R^2+r^2-2Rr\cos(\theta)}, \tag{5}$$

$$K_0(\chi)=\sum_{m=-\infty}^{\infty}K_m(R)I_m(r)\cos(m\theta). \tag{6}$$

After some manipulation $H(x;x')$ is found to be

$$H(x;x')=\frac{2}{\pi}\sum_{m=0}^{\infty}(2-\delta_{0,m})\cos(m\theta)\times\int_0^{\infty}\frac{K_m(k)I_m(kr')I_m(kr)\cos(kz)}{I_m(k)}\,\mathrm{d}k. \tag{7}$$

Here, $\delta_{0,m}$ is the Kronecker delta and is equal to unity for $m=0$, and zero otherwise. The validity of Eq. (7) can be demonstrated in several ways. Begin by noting that each term in Eq. (7) is a solution to the Laplace equation inside the cylinder. Then, setting $r=r_0=1$, gives

$$H(r_0,\theta,z;r',0,0)=\frac{2}{\pi}\sum_{m=0}^{\infty}(2-\delta_{0,m})\cos(m\theta)\times\int_0^{\infty}K_m(k)I_m(kr')\cos(kz)\,\mathrm{d}k, \tag{8}$$

which should equal Eq. (2). The integral in Eq. (8) is given explicitly in Gradshteyn and Ryzhik's[9] Eq. 6.672.4,

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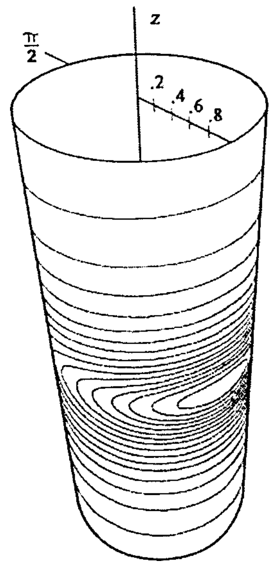

FIG. 1. Plot of Eq. (2) [$\Phi(r_0,\theta,z;r',0,0)$].

$$\int_0^\infty K_m(k)I_m(kr')\cos(kz)\mathrm{d}k$$

$$=\frac{1}{2\sqrt{r'}}Q_{m-1/2}\left(\frac{1+r'^2+z^2}{2r}\right). \tag{9}$$

Here $Q_m(x)$ is the Legendre function of the second kind of order m. When m is a half integer, $Q_{m-1/2}(x)$ is closely related to toroidal functions. In a recent paper, Cohl *et al.*[10] showed that the sum that results from substituting Eq. (9) into Eq. (8) is indeed equal to the right-hand side of Eq. (2). The full dimensionality can be readily recovered by substitutions $\theta\rightarrow\theta-\theta'$ and $z\rightarrow z-z'$.

Several interesting things follow from Eq. (7). First, the individual terms in the sum generally do *not* go to zero at the boundary when it is substituted into Eq. (1). Rather, the boundary condition is met by the entire sum. Second, the integrals over k generally behave well except when both r and r' are large. They do converge, but not quickly. Also, as one would expect, for large r', many terms in m are needed. Third, if the field point, r, is set equal to the source point, r', in Eq. (7), the result can be expanded term by term in a Taylor series in r. Then collecting terms of each order, the resulting integrals can be performed numerically to get the ''pseudopotential'' given by Tinkle and Barlow.[3]

$$H_{\text{pseudo}}(r)=-0.87969-1.0027r^2-1.0009r^4-1.0003r^6 -1.0001r^8\ldots \tag{10}$$

Note that Tinkle and Barlow[3] were unable to find the zeroth order term by their method. Finally, evaluating the fields from the gradient of the potential does not seen to give rise to convergence problems.

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