Simple theory of a nonlinear diocotron mode

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A simple perturbation theory of the frequencies and density distortions of the $l=1$ diocotron mode on pure electron plasma columns is presented. This mode is the dynamical state of an off-axis electron column which is orbiting around the containment axis at the $\mathbf{E}\times\mathbf{B}$ drift velocity. The theory predicts shifts in the orbit frequency and quadrupole distortions of the plasma surface, both varying as the offset squared, in good agreement with experiments. The results also apply to a two-dimensional vortex patch in an inviscid fluid with circular boundaries.

I. INTRODUCTION

Electron columns contained inside conducting cylinders in an axial magnetic field support electrostatic modes varying as $\exp(il\theta - i\omega t - ik_z z)$, known as *diocotron* modes. In this paper the diocotron mode with $l=1$ and $k_z=0$ will be discussed. This mode is simply a displacement of the entire column off the cylindrical axis. In this case, the column orbits around the cylindrical axis as well as rotating about its own center. The $l=1$ mode has proven to be fundamental to the understanding and manipulation of magnetized charged-particle systems, including electron beams,[1] magnetrons,[2] Penning traps,[3] and cylindrical electron plasma traps.[4]

For the experiments considered here,[5] the individual electron bounce time along the magnetic field is typically much shorter ($\tau_b \sim 1\ \mu$sec) than the wave period ($2\pi/\omega \sim 20\ \mu$s), and the cyclotron orbit period is much shorter than either ($\tau_c \sim 0.01\ \mu$sec). The axial bouncing of the individual electrons averages over any z variations at a rate fast compared to r-θ motions, and the density is uniform in z except for a small Debye sheath at the ends. Furthermore, the electron gyroradius is negligible compared to the radius of the column, and modes with $k_z \neq 0$ are damped and not normally observed unless externally launched. Therefore, to a good approximation, an electron can be approximated as a point in the r-θ plane governed by the two-dimensional (2-D) drift-Poisson equations. The velocity in the r-θ plane is given by $\bar{u}=(c/B)\bar{E}\times\hat{z}$, and an electron column spins around its axis due to the self-electric field. This gives rise to a vorticity $\tilde{n}=(4\pi ec/B)n$, where n is the density of electrons (in cm^{-3}) and B is the magnetic field.

The analysis presented here also applies to the 2-D motion of a vortex patch in an inviscid fluid contained within a cylindrical tank. This is due to the isomorphism between the 2-D drift-Poisson equations and the 2-D Euler equations governing a constant density inviscid fluid.[6] Thus, an initial distribution of fluid vorticity $\tilde{n}(r,\theta,t=0)$ in a cylinder will evolve the same as the same vorticity distribution in the electron column.

If a circular electron column is offset from the cylindrical wall axis by an amount D, the electric field from redistributed wall charges causes the column to drift around the cylindrical axis in addition to the faster rotation of the column around its own axis. To order D/R_w, where R_w is the wall radius, a circular column remains circular as it orbits.

This paper presents a simple water-bag (constant density) theory of the shape and frequency of the column to order $(D/R_w)^2$. At this order, the wall charges produce a straining field in the column that cause the column to distort into an oval shape; this in turn induces more wall charges that act to reduce the strain. The distortion is calculated using an equilibrium equation for constant vorticity patches,[7] and the distortion and orbit frequency are expressed as functions of D and R_p, where R_p is the patch radius. The theory predicts quadrupole density distortions and frequency shifts proportional to $(D/R_w)^2$, in good agreement with a previous experiment.[5]

II. BASIC GEOMETRY AND EQUATIONS

The geometry is shown in Fig. 1, which depicts a constant density, offset column orbiting the cylinder axis with frequency ω. The coordinate system (X_c, Y_c) has the cylindrical wall axis as its origin, while the (X,Y) system has its origin at the column center, so that $X_c = X + D$. The direction of Y has been chosen so that it points along the long axis of the distorted column. For convenience we use scaled coordinates $x \equiv X/R_w$ and $y \equiv Y/R_w$, with a similar definition for (x_c, y_c). In addition, we define the normalized radius $r_p \equiv R_p/R_w$ and displacement $d \equiv D/R_w$.

For a circular column, the wall charges are equivalent to a point image charge at a radius R_w/d in the direction of the offset, with charge per unit axial length of $+Ne$, where the column has charge per length $-Ne$, and $N = n\pi R_p^2$. The electric potential inside the circular column is given by

$$\phi = Ne[-\ln(b^2 + r^2 - 2xb) + 2\ln\ r_p - 1 + r^2/r_p^2], \quad (1)$$

where $b \equiv 1/d - d$ and $r^2 \equiv x^2 + y^2$. The first term in Eq. (1) is the potential of the positive image charge, and the last terms are the self-potential of the column. Differentiating Eq. (1), and expanding in powers of d, the electric field in the circular column is found to be

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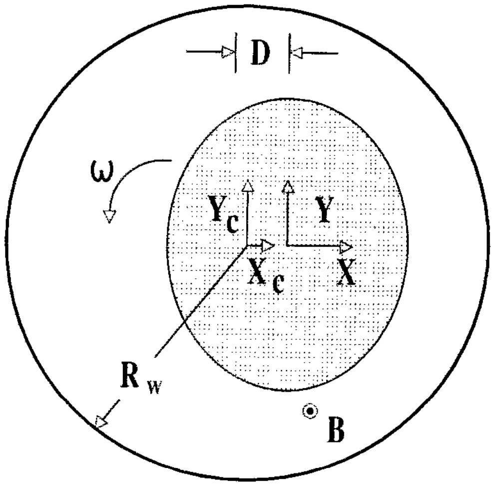

FIG. 1. Geometry of the $l=1$ diocotron mode at large amplitude. Here an orbiting electron column is shown distorted into an oval cross section.

$$E_x^c=-\frac{2Ne}{R_w}\left(d+xd^2+(1+x^2-y^2)d^3+\cdots+\frac{x}{r_p^2}\right),$$
$$E_y^c=-\frac{2Ne}{R_w}\left(-yd^2-2xyd^3+\cdots+\frac{y}{r_p^2}\right). \tag{2}$$

Again, the last term is the self-field, which causes the column to spin about its own axis. The other terms are from the image, which to order d^1 are constant over the column. These fields cause the column to orbit around the cylinder axis while maintaining a circular shape. In order d^2, the image field produces a "straining" field at the column, and a circular column is not in equilibrium.

To be in equilibrium the boundary shape must not change with time in the rotating frame. Since electrons drift around potential contours, this implies that the boundary shape must match a potential contour in the rotating frame. The second-order image field distorts these potential contours from being circular, and therefore distorts the column shape. We model this distortion as a quadrupole charge distribution on the surface of the column, as

$$\frac{\delta N(\theta)}{N}=-q_2\cos 2\theta\,\frac{d\theta}{2\pi}. \tag{3}$$

Here, δN is the number of surface charges in angular extent $d\theta$. Note that the distortion is described in the frame (x,y) centered on the electron column. This is an improvement over a previous nonlinear perturbation theory of this diocotron mode,[8] which uses the cylinder axis as the coordinate center. In the (x_c,y_c) frame, large perturbations would be needed just to model the column offset.

The quadrupole surface charge on the column causes a redistribution of wall charges. The calculation of the field due to the quadrupole image is somewhat tedious, but can be simplified using the method of inversion.[9] This technique implies that in 2-D, the image potential ϕ_i due to wall charges can be calculated from the potential ϕ with no wall at all by $\phi_i(r_c,\theta_c)=-\phi(1/r_c,\theta_c)-Q\ln r_c^2$, where the subscript c refers to a coordinate system centered on the conducting cylinder, and the net charge per length in the cylinder is Q. Assuming the surface charge of Eq. (3) lies on a circle of radius r_p, the field outside the column due to the quadrupole surface charge with no wall is

$$\phi^q(r,\theta)=\frac{1}{2}q_2Ner_p^2\frac{\cos 2\theta}{r^2}$$
$$=\frac{1}{2}q_2Ner_p^2\left[\frac{r_c^2\cos 2\theta_c-2dr_c\cos\theta_c+d^2}{(r_c^2+d^2-2dr_c\cos\theta_c)^2}\right], \tag{4}$$

where (r,θ) are centered on the column. The method of inversion gives

$$\phi_i^q(r_c,\theta_c)=-\frac{1}{2}q_2Ner_p^2\left(\frac{\cos 2\theta_c-2dr_c\cos\theta_c+d^2r_c^2}{(1+d^2r_c^2-2dr_c\cos\theta_c)^2}\,r_c^2\right), \tag{5}$$

since $Q=0$ for the quadrupole surface charge. From Eqs. (4) and (5), the electric field due to the quadrupole charge and its image is

$$E_x^q=-\frac{2Ne}{R_w}\left(-x\frac{1}{2}q_2r_p^2-\frac{1}{2}q_2\,dr_p^2\right.$$
$$\left.-(x^2-y^2)\frac{3}{2}q_2\,dr_p^2+\cdots+\frac{1}{2}q_2\frac{x}{r_p^2}\right),$$
$$E_y^q=-\frac{2Ne}{R_w}\left(y\frac{1}{2}q_2r_p^2+2xy\frac{3}{2}q_2\,dr_p^2+\cdots-\frac{1}{2}q_2\frac{y}{r_p^2}\right). \tag{6}$$

These equations are to order d^3, since we will find q_2 is order d^2 in the next section. The first terms in Eq. (6) are from the quadrupole image charges, which act to reduce the strain from the point image, and the last term is from the self-field. The total electric field [the sum of Eqs. (2) and (6)] describes the velocity field in the column, since $u_x=cE_y/B$ and $u_y=-cE_x/B$.

III. EQUILIBRIUM SHAPE AND FREQUENCY

The column must have a shape that is in equilibrium in this velocity field. Moore and Saffman[7] found that a constant vorticity patch with an elliptical boundary is an exact equilibrium solution in an external velocity field $\bar{u}=(u_x,u_y)$ of the form

$$u_x=sY,\quad u_y=sX, \tag{7}$$

where s is the straining rate. (This result is a generalization of Kirchhoff's rotating elliptical vortex.[10]) Furthermore, they derived a relationship between the boundary shape and the strain rate and vorticity of the patch. The boundary is parametrized by $\gamma\equiv a/b$, the ratio of the major to minor axes of the ellipse, which is related to the quadrupole moment by $\gamma^2\approx 1+q_2$, to first order in q_2. In terms of the symbols used in this paper, the equilibrium of Ref. 7 is specified by

$$s(\gamma+1)/(\gamma-1)+\omega=(\tilde{n}-2\omega)\gamma/(\gamma^2+1). \tag{8}$$

Here, the vorticity patch is stationary in a frame rotating at a rate ω around the cylinder axis, and the straining field is also stationary in this frame. The vorticity, $\tilde{n}$, is measured in the lab frame. Expressing this relation in terms of the quadrupole moment and linearizing in q_2, the equilibrium equation becomes

$$2s=(\tfrac{1}{2}\tilde{n}-2\omega)q_2\,. \tag{9}$$

The frequency ω and straining rate s can be calculated from the velocity field. The orbit frequency is

$$\omega=\frac{u_y}{D}=-\frac{cE_x}{DB}=\omega_1\left(1+d^2-\frac{1}{2}\,q_2r_p^2\right), \tag{10}$$

where the nonconstant terms in the velocity average to zero over the column. Here, $\omega_1\equiv 2Nec/BR_w^2$, and is identical to the frequency found in linear theory.[11] The orbit frequency is different from ω_1 due to two effects: an increase in frequency by $\omega_1 d^2$ due to the point image being closer to the column than linear theory assumes, and a decrease in frequency by $-\frac{1}{2}\omega_1 q_2 r_p^2$ due to a reduction in the electric field from the spreading of charges in the θ direction as the column distorts.

The external straining rate can be found from the image velocity field, that is, eliminating the self-field in Eqs. (2) and (6). The strain is

$$s=\frac{\partial u_y}{\partial X}=-\frac{c}{BR_w}\,\frac{\partial E_{x,i}}{\partial x}=\omega_1\left(d^2-\frac{1}{2}\,q_2r_p^2\right), \tag{11}$$

where higher-order terms are neglected. If Eqs. (10) and (11) are substituted into Eq. (9), the quadrupole moment can be expressed in terms of d and r_p. If q_2 were order d^1, then the only solution is $q_2=0$. Assuming q_2 is order d^2, we find

$$q_2=\frac{2r_p^2}{(1-r_p^2)^2}\,d^2. \tag{12}$$

If Eq. (12) is substituted into Eq. (10), the frequency shift is found to be

$$\frac{\omega-\omega_1}{\omega_1}=\frac{1-2r_p^2}{(1-r_p^2)^2}\,d^2, \tag{13}$$

where higher-order terms have been neglected in both Eqs. (12) and (13).

IV. COMPARISON TO EXPERIMENT

In a previous experiment,[5] the quadrupole moment of an electron column was measured for a range of column radii and offsets. The experiments were with approximately constant density columns, so the results of the water-bag approximation used here should be relevant. The normalized quadrupole moment q_2, for a 2-D charge distribution $\rho(x,y)$, can be calculated from

$$q_2\equiv\frac{\int(y^2-x^2)\rho(x,y)dx\,dy}{\int(x^2+y^2)\rho(x,y)dx\,dy}, \tag{14}$$

where the (x,y) coordinate system is as shown in Fig. 1, with the y axis lying along the major axis of the column. If

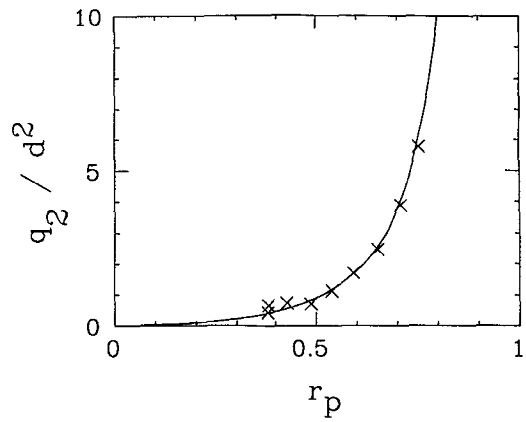

FIG. 2. Measured quadrupole distortion over d^2 versus column radius compared to Eq. (12) (solid line).

an initially circular column of radius R_p is elongated by a small amount Δ along the y axis, and shortened by Δ along the x axis, then $q_2\approx\Delta/R_p$.

In the experiment, q_2 was measured as a function of the offset d. The measured q_2 was found to vary approximately as d^2 up to the largest measured offsets ($d\sim 0.3$). In addition, each measured q_2 was divided by the measured d^2 and averaged together for each radius column. The average value of q_2/d^2 is plotted versus column radius in Fig. 2. The prediction of Eq. (12) is plotted as a solid line. It can be seen that there is good agreement, so that Eq. (12) is a good fit to the data over the entire experimental range.

The experiments also measured the mode frequency versus column offset, $\omega(d)$, which defined a small-amplitude limit $\omega_0\equiv\omega(d=0)$. In all cases, we found $\Delta\omega\equiv\omega(d)-\omega_0\propto d^2$, as predicted by Eq. (13). In Fig. 3, we plot the experimental data of Fig. 5 of Ref. 5 as $(\omega-\omega_0)/\omega_0 d^2$. The curve in Fig. 3 is $(\omega-\omega_1)/\omega_1 d^2$ from Eq. (13). Note that the theory predicts a *negative* frequency change for $r_p>1/\sqrt{2}$, and this was observed for the largest radii of $d=0.76$.

It should be pointed out that in the experiment, ω_0 is the measured small-amplitude limit, which is about 10%

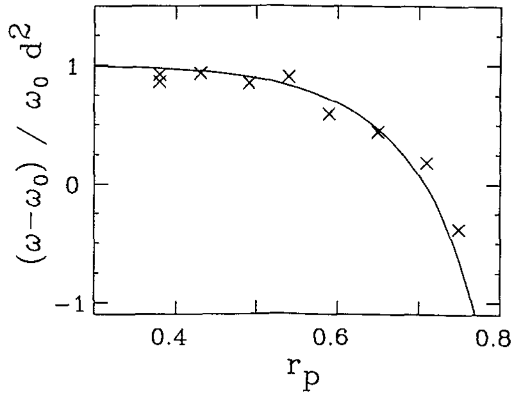

FIG. 3. Measured fractional frequency shift over d^2 versus column radius compared to Eq. (13) (solid line).

higher than the predicted linear frequency ω_1. This is due to the effect of the finite length of the column. Essentially, the electric containment fields at the ends have a radial component which gives an increase in the column-averaged angular drift frequency ω. At present, there is no theory of this effect at large amplitude. Presumably, the finite length frequency shift $\omega_0 - \omega_1$ also varies with d, so the comparison of Fig. 3 may be inaccurate by about 10%.

V. SUMMARY

In summary, a simple perturbation theory of the nonlinear $l=1$ diocotron mode for electron columns was described that is in good agreement with measured distortions and frequency shifts. The theory also applies to a vortex patch in an inviscid fluid interacting with a circular wall.

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