Vacuum-polarization corrections to solar-fusion rates

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The vacuum-polarization (VP) corrections to rates for nuclear-fusion reactions in the *pp* chain and in the CNO cycle are calculated. For the reactions of particular importance to the solar-neutrino problem, the ^{3}He(^{3}He,2p)^{4}He, ^{3}He(α,γ)^{7}Be, ^{7}Be(p,γ)^{8}B, and ^{14}N(p,γ)^{15}O reactions, we find the magnitude of the effect to be less than 2%. The effect of VP on all the other reaction rates is expected to be of a similar order of magnitude. We discuss how these results affect the predicted fluxes of solar neutrinos.

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The cross sections for the nuclear reactions in the *pp* chain and in the CNO cycle are important for stellar-evolution calculations of main-sequence stars [1] and, in particular, for the calculated flux of solar neutrinos [2]. The energies at which these cross sections are measured in the laboratory are generally higher than the energies of interest in the Sun. The cross sections at solar energies are determined by extrapolating the measurements to lower energies using the Gamow penetration factor, which expresses the probability of quantum tunneling through the Coulomb barrier. In this paper, we calculate the effect of vacuum-polarization (VP) corrections to the electrostatic potential on the rates for nuclear reactions in the *pp* chain and in the CNO cycle. We show that such corrections lead to only small changes in the predicted flux of solar neutrinos.

In addition to the Coulomb potential, $V_C=e^2/r$, there is an additional contribution to the electrostatic potential, the Uehling potential, $V_{\rm VP}(r)$ [3], which arises from quantum corrections. The complete electrostatic potential for two nuclei with charges Z_1 and Z_2 separated by a distance r is

$$V(r)=V_C+V_{\rm VP}=\frac{Z_1Z_2e^2}{r}+\frac{Z_1Z_2e^2}{r}\left[\frac{2\alpha I(r)}{3\pi}\right], \quad (1)$$

where α is the fine-structure constant, and

$$I(r)=\int_1^\infty e^{-2m_erx}\left[1+\frac{1}{2x^2}\right]\frac{(x^2-1)^{1/2}}{x^2}dx , \quad (2)$$

where m_e is the electron mass. The function $I(r)$ has the limiting forms

$$I(r)=-\gamma-\tfrac{5}{6}-\ln(m_er) \quad \text{for } m_er\ll 1 \quad (3)$$

and

$$I(r)=\frac{3(2\pi)^{1/2}}{4}\frac{-2m_er}{(2m_er)^{3/2}} \quad \text{for } m_er\gg 1 . \quad (4)$$

The function $I(r)$ has a logarithmic singularity for very small radii, and exhibits an exponential falloff (arising from the exchange of a virtual electron-positron pair) for $r\gtrsim 1/2m_e$. The probability for tunneling through the electrostatic barrier is affected by the presence of the Uehling potential [4].

The energy dependence of a nonresonant fusion cross section is ordinarily written (see, e.g., Refs. [2,5])

$$\sigma(E)\equiv\frac{S(E)}{E}\Gamma^{(0)}(E) , \quad (5)$$

where $S(E)=S(0)+S'(0)E$ is a slowly varying function of E, and $S(0)$ and $S'(0)$ are determined by fits to experimental measurements of the cross section. The quantity $\Gamma^{(0)}(E)$ is the usual Gamow penetration factor, the probability of tunneling through the Coulomb barrier. In practice, $S(E)$ is deduced by laboratory measurements of the cross section at energies of order 100 keV to several MeV, and the cross section is then extrapolated to energies, O(10 keV), typical of solar reactions, through Eq. (5). In this paper, we use WKB approximation similar to that used by Gould [4] to estimate the vacuum-polarization corrections to the Gamow penetration factor for any binary reaction.

For a pure Coulomb potential, the Gamow penetration factor for reacting nuclei with center-of-mass energy E is simply

$$\Gamma^{(0)}(E)=\exp\left[-\frac{2}{\hbar}\int_0^b[2\mu(V_C-E)]^{1/2}dr\right]=\exp(-2\pi\eta) , \quad (6)$$

where $b=Z_1Z_2e^2/E$ is the turning point radius, which is determined by $V_C(b)=E$. Here, $\eta=Z_1Z_2e^2/\hbar v$, where $v=(2E/\mu)^{1/2}$ is the relative velocity of the incoming nuclei. Also, $\mu=M_pA_1A_2/(A_1+A_2)$ is the reduced mass of the system, and A_1 and A_2 are the atomic mass numbers of the reacting nuclei.

To include vacuum polarization in the cross section, we make the substitution

$$\Gamma^{(0)}(E)\rightarrow\Gamma(E) \quad (7)$$

in Eq. (5), where $\Gamma(E)$ is the probability of tunneling through the complete electrostatic potential. Now,

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$$\Gamma(E)=\exp\left[-\frac{2}{\hbar}\int_0^{b_{\mathrm{VP}}}[2\mu(V_C+V_{\mathrm{VP}}-E)]^{1/2}dr\right]$$
$$\equiv\Gamma^{(0)}(E)[1-\Delta(E)]\ , \qquad (8)$$

where the turning-point radius is now given by $V_C(b_{\mathrm{VP}})+V_{\mathrm{VP}}(b_{\mathrm{VP}})=E$ and can be written

$$b_{\mathrm{VP}}=b[1+2\alpha I(b)/3\pi]\ .$$

We expand the integral in powers of the fine-structure constant, α, and use the result for b_{VP} to find

$$\Delta(E)=\frac{4\alpha\eta}{3\pi}\int_0^1\frac{I(bx)}{\sqrt{x-x^2}}dx\ . \qquad (9)$$

The function $\Delta(E)$ is plotted in Fig. 1 for the ^{3}He(^{3}He,$2p$)^{4}He (solid line), ^{3}He(α,γ)^{7}Be (short-dash line), ^{7}Be(p,γ)^{8}B (long-dash line), and ^{14}N(p,γ)^{15}O (dot-dash line) reactions. The function $\Delta(E)$ initially increases from its value at $E=0$ with increasing energy, reaches a maximum at an energy $E_{\max}$ and then decreases roughly as $E^{-1/2}$ at higher energies. Although the lower limit of $\Delta(E)$ is very conservatively given by 0 (at very large energies), the lower limit of $\Delta(E)$ for energies at which measurements are performed is actually not much smaller than $\Delta(E_{\max})$.

The most probable energy of interaction for nuclei in the core of the Sun is [2]

$$E_{\odot}=1.22[Z_1^2Z_2^2(\mu/m_p)T_6^2]^{1/3}\ \mathrm{keV}\ , \qquad (10)$$

where T_6 is the temperature in units of 10^6 K. We take $T_6=14$ (the temperature at which energy production is maximized). To self-consistently determine the effect of vacuum polarization on reaction rates in the Sun, the VP correction must be included in the analysis of the data from which the low-energy cross sections are extrapolated. This is done by fitting the data to Eq. (5) using Γ instead of $\Gamma^{(0)}$. The cross section is then evaluated at energies typical of nuclear reactions in the Sun with the $S(E)$ obtained and with $\Gamma(E)$ rather than $\Gamma^{(0)}(E)$. We then determine the effect of VP by comparing this result for the cross section with the standard result.

For the ^{3}He(^{3}He,$2p$)^{4}He reaction, we used data from Ref. [6] and found that including VP self-consistently in the entire analysis decreases the reaction rate by about 0.2%. For the ^{3}He(α,γ)^{7}Be reaction, we used the Parker-Kavanagh data set [7] and found that VP decreases the reaction rate by 1.6%.[1] The decrease is due to the fact that the data set is weighted at higher energies where the VP correction is smaller than that at solar energies. The magnitude of our result is also much smaller than 4.6%, which was obtained in Ref. [4]. The discrepancy is due to the fact that we included VP in fitting the data, whereas Gould evaluated the correction to the cross section at solar energies without correcting for the effect of VP on the data. In addition, our value of $\Delta(E_{\odot})=5.3\%$ differs from Gould's value (4.6%) because Gould evaluated the correction to the velocity factor in the denominator of the WKB approximation to the wave function. In the standard treatment, the Gamow penetration factor includes only the leading exponential factor in the WKB approximation. Therefore, for consistency with the standard experimental procedure, we have not included the velocity factor. For the ^{7}Be(p,γ)^{8}B reaction, we used the data set of Kavanagh *et al.* [8], and found that VP decreased the reaction rate by 0.1%. In order to obtain this result, we used only data points at energies below the resonance. This is consistent with the treatment in the most recent analysis of the ^{7}Be(p,γ)^{8}B $S(0)$ factor [9]. The ^{14}N(p,γ)^{15}O reaction is the slowest process in the CNO cycle, so the rate for production of neutrinos from ^{13}N and ^{15}O decays is controlled by the rate for this reaction. The best estimate of the astrophysical S factor for this reaction comes from a data set made up of measurements from 0.2 to 3.6 MeV [10]. We used a simulated data set made up of 18 uniformly spaced data points spread over this energy range and found that VP decreased the solar reaction rate by 0.8%. For each of these four reactions, the effect of VP is much smaller than the uncertainty in the measured value of $S(0)$.

Our data sets differ slightly from those used to obtain the current best estimates for the low-energy cross-section factors. Our results on the effect of VP would be altered by negligible amounts if the exact data sets were used.

For the other reactions in the pp chain and in the CNO cycle, the effect of including VP in the entire analysis can be crudely estimated by comparing the relative magnitudes of the VP correction, $\Delta(E)$, at solar energies to that at energies at which the measurements are performed. Of course, measurements are performed over some range of energies, and the VP correction may change considerably over this range, but conservative upper and lower bounds on the VP correction can be provided. We do so by comparing $\Delta(E)$ at solar energies with conservative estimates of the maximum and minimum VP corrections to the data points.

In Table I, we list the VP correction to the Gamow penetration factor at solar energies, $\Delta(E_{\odot})$, the most probable energy of interaction, $E_{\odot}$, the maximum of $\Delta(E_{\max})=\Delta(E)$, and the energy, $E_{\max}$ at which this occurs. In addition, we have listed the best estimates, δ_{VP}, of the effect of VP on the rates for the reactions that we have analyzed more carefully. In all cases, $\Delta(E_{\max})$ is only slightly larger than $\Delta(E_{\odot})$, and $E_{\max}>E_{\odot}$. For each reaction, the effect of VP decreases the reaction rate by no more than $\Delta(E_{\odot})$ (which is always much less than 6%), and may increase the reaction rate by no more than $\Delta(E_{\max})-\Delta(E_{\odot})$ (which is always less than 1%). The effect of inclusion of VP in the entire analysis should always be much smaller than $\Delta(E_{\odot})$. This is because $\Delta(E_{\max})$ usually occurs near the lowest energies at which measurements are performed, and $\Delta(E)\sim E^{-1/2}$ remains non-negligible even at higher energies (see Fig. 1). This is illustrated by the results for the reactions that we have studied more carefully. Therefore, VP should generally

[1]We are grateful to P. Parker for providing the ^{3}He(α,γ)Be and ^{7}Be(p,γ)^{8}B data sets.

TABLE I. Vacuum-polarization corrections to nuclear-reaction rates. $E_\odot$ and E_{max} are in keV, and $\Delta(E)$ and δ_{VP} are in percentages.

VP corrections to nuclear-reaction rates					
Reaction	$E_\odot$	$\Delta(E_\odot)$	E_{max}	$\Delta(E_{max})$	δ_{VP}
$p+p\to{}^2\mathrm{H}+e^++\nu_e$	6	1.4	11	1.5	−0.6
${}^3\mathrm{He}+{}^3\mathrm{He}\to\alpha+2p$	20	5.0	46	5.0	−0.2
${}^3\mathrm{He}+{}^4\mathrm{He}\to{}^7\mathrm{Be}+\gamma$	21	5.3	46	5.4	−1.6
${}^7\mathrm{Li}+p\to2\alpha$	14	3.3	31	3.3	
${}^7\mathrm{Be}+p\to{}^8\mathrm{B}+\gamma$	17	3.8	46	3.8	−0.1
${}^3\mathrm{He}+p\to{}^4\mathrm{He}+e^++\nu_e$	10	2.5	21	2.5	
${}^{12}\mathrm{C}(p,\gamma){}^{13}\mathrm{N}$	23	4.8	66	4.8	
${}^{13}\mathrm{C}(p,\gamma){}^{14}\mathrm{N}$	23	4.8	66	4.9	
${}^{14}\mathrm{N}(p,\gamma){}^{15}\mathrm{O}$	25	5.2	76	5.3	−0.8
${}^{15}\mathrm{N}(p,\gamma){}^{16}\mathrm{O}$	25	5.2	76	5.3	
${}^{15}\mathrm{N}(p,\alpha){}^{12}\mathrm{C}$	25	5.2	76	5.3	
${}^{16}\mathrm{O}(p,\gamma){}^{17}\mathrm{O}$	28	5.5	91	5.6	

have no more than an $O(1\%)$ effect on nuclear-reaction rates in the Sun.

The rate for the initial $p+p\to{}^2\mathrm{H}+e^++\nu_e$ reaction is also affected by vacuum polarization [4,11]. The cross section is too small for this reaction to be observed in the laboratory, so the rate for the solar *pp* reaction is calculated instead of extrapolated from measurements at higher energies. Therefore, the effect of VP on the *pp* rate is determined in a different manner, and the result is that VP decreases the rate by 0.6% [11]. We list this best estimate, as well $\Delta(E_\odot)$ and $\Delta(E_{max})$, in the table.

Our results imply that VP will have little effect on the predicted flux of solar neutrinos. The flux of neutrinos from ^{8}B decay depends most sensitively on the nuclear-reaction rates. The dependence of the ^{8}B neutrino flux may be written [2]

$$\phi({}^8B)\propto S_{11}^{-2.6}S_{33}^{-0.40}S_{34}^{0.81}S_{17}^{1.0}\ , \tag{11}$$

where S_{11}, S_{33}, S_{34}, and S_{17} are the low-energy cross-section factors for the *pp*, ${}^3\mathrm{He}({}^3\mathrm{He},2p){}^4\mathrm{He}$, ${}^3\mathrm{He}(\alpha,\gamma){}^7\mathrm{Be}$, and ${}^7\mathrm{Be}(p,\gamma){}^8\mathrm{B}$ reactions, respectively. Inserting our best estimates for the effect of VP on the low-energy cross-section factors, we find that inclusion of VP in the analysis increases the predicted flux of ^{8}B neutrinos by only 0.2%.

The dependence of the neutrino fluxes from the other reactions in the *pp* chain and CNO cycle on small changes in the low-energy cross-section factors are similarly determined [2]. The flux of ^{7}Be neutrinos will be decreased by 0.7% by VP corrections to the nuclear-reaction rates; the neutrino fluxes from ^{13}N and ^{15}O decay will be increased by 0.8% and 1.0%, respectively, and the flux of *pp* neutrinos will be increased by 0.02%. These result in an increase of 0.01 solar-neutrino units (SNU) in the predicted event rate for the chlorine experiment and a decrease of 0.11 SNU in the predicted event rate for the gallium experiments.

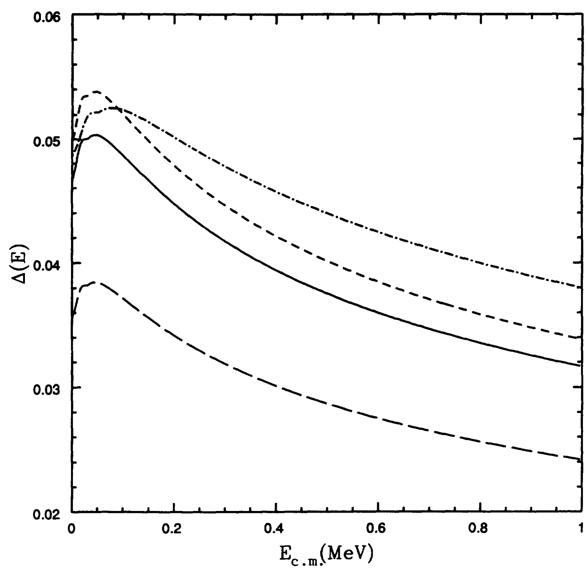

FIG. 1. Plot of $\Delta(E)$ versus the center-of-mass energy. The solid line is for the ${}^3\mathrm{He}({}^3\mathrm{He},2p){}^4\mathrm{He}$ reaction, the short-dash line is for the ${}^3\mathrm{He}(\alpha,\gamma){}^7\mathrm{Be}$ reaction, the long-dash line is for the ${}^7\mathrm{Be}(p,\gamma){}^8\mathrm{B}$ reaction, and the dot-dash line is for the ${}^{14}\mathrm{N}(p,\gamma){}^{15}\mathrm{O}$ reaction.

In summary, we have evaluated the effect of vacuum polarization on the determination of nuclear-fusion reaction rates in the Sun. We find that the magnitude of the VP correction is never more than 2% for the reactions that are most important for solar-neutrino calculations. The VP correction has only a small effect on the calculated solar-neutrino fluxes.

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